Solutions, Extraneous Solutions, and the Logic of Solving Equations

Josh Chesler

California State University, Long Beach

CMC-S, Oct. 24, 2014

Motivation

- Textbooks may have statements like:
  - “When you multiply each side of an equation by the LCD [Least Common Denominator], you may get an extraneous solution. An **extraneous solution** is a solution to a resulting equation that is not a solution to the original equation. Because of extraneous solutions, it is especially important to check your answers.” (p. 675)
- This is a statement ***that*** we may get extraneous solutions. But ***why***? ***how***?
- As is appropriate, the textbook provides an example immediately following that statement….

Burger, Edward B.; Chard, David J.; Hall, Earlene J.; Kennedy Paul A.; Leinwand, Steven J.; Renfro, Freddie L.; Seymour, Dale G.; Waits, Bert K. (2008) *Holt Algebra 1 California Teacher Edition*. Holt, Reinehart and Winston.

The example…

3 Extraneous Solutions

Solve $\frac{x-9}{x^2-9} = \frac{-3}{x-3}$. Check your answer.

$(x-9)(x-3) = -3(x^2-9)$	*Use cross products.*
$x^2 - 12x + 27 = -3x^2 + 27$	*Multiply the left side. Distribute −3 on the right side.*
$4x^2 - 12x + 27 = 27$	*Add $3x^2$ to both sides.*
$4x^2 - 12x = 0$	*Subtract 27 from both sides.*
$4x(x-3) = 0$	*Factor the quadratic expression.*
$4x = 0$ or $x - 3 = 0$	*Use the Zero Product Property.*
$x = 0$ or $x = 3$	*Solve for x.*

Check

$\frac{x-9}{x^2-9} = \frac{-3}{x-3}$

$\frac{0-9}{0^2-9}$	$\frac{-3}{0-3}$
$\frac{-9}{-9}$	$\frac{-3}{-3}$
1	1 ✓

$\frac{x-9}{x^2-9} = \frac{-3}{x-3}$

$\frac{3-9}{3^2-9}$	$\frac{-3}{3-3}$
$\frac{-6}{0}$	$\frac{-3}{0}$

Because both $\frac{-6}{0}$ and $\frac{-3}{0}$ are undefined, 3 is not a solution.

3 is an extraneous solution. The only solution is 0.

- "When you multiply each side of an equation by the LCD, you may get an extraneous solution."
- In the first step, both sides are multiplied by $(x^2-9)(x-3)$, which is NOT the LCD of (x^2-9) and $(x-3)$.
- Focusing on the LCD is a red herring. We'll see why.
- There are simpler, less specific ideas that are more useful and can help us explain ***why*** we may get extraneous solution.

CCSSM HS Algebra Standards

- SMP 3: Construct Viable Arguments and Critique the Reasoning of Others
- SMP 6: Attend to Precision

Reasoning with Equations and Inequalities A-REI

Understand solving equations as a process of reasoning and explain the reasoning

1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

CCSSM Expressions & Equations Domain – Grade 6

- Cluster: Reason about and solve one-variable equations and inequalities.
- 6.EE.B.5 **Understand** solving an equation or inequality **as a process of answering a question**: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Our Plan

1. What is a solution?
2. Equivalent equations & extraneous solutions
3. The logic of solving equations
4. The logic of solving systems of equations (time permitting)

Throughout, we'll make connections to the CCSSM and SBAC.

NOTE: *The context of what follows is real (not complex) solutions to equations.*

What is a solution?

Grade 11 SBAC

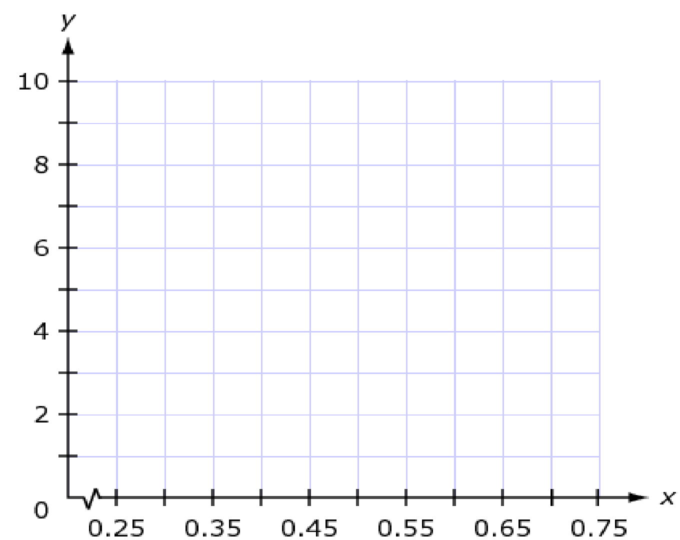
What does a solution look like for an equation of two variables?

How many solutions are there?

An equation is shown.

$$y = \frac{3}{\sqrt{x}}$$

Use the Add Point tool to plot three solutions to this equation on the coordinate grid.



It’s always the same question…

$3x = 6$	What are all the values of x which make this true?
$x^2 + 4x + 3 = 0$	What are all the values of x which make this true?
$x^2 + 4x + 3 < 0$	What are all the values of x which make this true?
$3x = y$	What are all the pairs (x, y) that make this true?
$3x - y = 0$	What are all the pairs (x, y) that make this true?
$x^2 + 4x + 3 < y$	What are all the pairs (x, y) that make this true?
$x^2 + y^2 = 1$	What are all the pairs (x, y) that make this true?
$3x + 2y + 3z = 1$	What are all the triples (x, y, z) that make this true?
$kx - y = 0$	What are all the pairs (x, y) that make this true? Or, is it all triples (k, x, y) which make the equation true? By convention, it’s typically the former.
$\begin{cases} x + y = 10 \\ x - y = 2 \end{cases}$	What are all the pairs (x, y) which make both equations true simultaneously?
$3x + 6 = 3(x + 2)$	What are all the values of x which make this true? If an equation in one variable is true for all values of the variable make it true, then the equation is an identity or property.

… and it's always a set.

Remember, a relation is a set of ordered pairs. Some relations are functions.

So we can ask the following questions about these equations of two variables:

1. Does this equation determine y as a function of x?
2. Does this equation determine x as a function of y?

$3x = 6$	The set of all x which make this true. $\{x \mid 3x = 6\}$
$x^2 + 4x + 3 = 0$	The set of all x which make this true.
$x^2 + 4x + 3 < 0$	The set of all x which make this true.
$3x = y$	The set of all (x, y) which make this true.
$3x - y = 0$	The set of all (x, y) which make this true.
$x^2 + 4x + 3 < y$	The set of all (x, y) which make this true.
$x^2 + y^2 = 1$	The set of all (x, y) which make this true.
$3x + 2y + 3z = 1$	The set of all (x, y, z) which make this true.
$kx - y = 0$	The set of all (x, y) which make this true. Or, is The set of all (x, y, k) which make this true. By convention, it's typically the former.
$\begin{cases} x + y = 10 \\ x - y = 2 \end{cases}$	The set of all (x, y) which make both equations true simultaneously.
$3x + 6 = 3(x + 2)$	The set of all x which make this true.

SBAC 8

- Drag numbers into the boxes to complete each equation with the given number of solutions.

> Think about the "process of answering a question: which values from a specified set, if any, make the equation or inequality true?"

0
1
2
3
4
5
6
7
8
9

A. Equation with no solutions.

$8x - 3x + 2 - x = \square \; x + \square$

B. Equation with one solution.

$8x - 3x + 2 - x = \square \; x + \square$

C. Equation with infinitely many solutions.

$8x - 3x + 2 - x = \square \; x + \square$

SBAC 11

SMP 7: Look For and Make Use of Structure

Given:

$$\sqrt{3x+1} - \sqrt{ax+b} = 0$$

Select values of a and b to complete the statements about the solutions to the given equation.

Choices: $-5, -4, -3, -2, -1, 0\ , 1, 2, 3, 4, 5$

a. The equation has no solutions when $a =$ ___ and $b =$ __

b. The equation has an infinite number of solutions when $a =$ __ and $b =$ __.

c. When $a = 1$ and $x = 2$, $b =$ __.

Equivalent Equations & Extraneous Solutions

Solving a Simple Equation

"Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method."

Here's a simple equation: $\frac{1}{2}x + 3 = 13$

We start with the assumption that that equation has a solution.

- Why?
  - Otherwise, it may be a false statement. There's not much we can do with that.
- What if that assumption was wrong?
  - You will reach a contradiction.
- In this example, we can we write "⇔" between each pair of lines.

$$\frac{1}{2}x + 3 = 13$$
$$\frac{1}{2}x = 10$$
$$x = 20$$

"⇒" means "implies"
"⇔" means "if and only if" or "is equivalent to/is the same as"

- So, $\frac{1}{2}x + 3 = 13 \Leftrightarrow \frac{1}{2}x = 10 \Leftrightarrow x = 20$. In other words, $\frac{1}{2}x + 3 = 13 \Leftrightarrow x = 20$. I can also read the implications from right to left: $x = 20 \Rightarrow \frac{1}{2}x + 3 = 13$.

Why are these equations equivalent?

$$\frac{1}{2}x + 3 = 13$$
$$\frac{1}{2}x = 10$$
$$x = 20$$

- A good way to think about it: All the steps we did are reversible.
  - Subtracting 3 is reversed by adding 3.
  - Dividing by ½ is reversed by multiplying by ½.
- So I can read these three lines from top to bottom, or bottom to top.

Equivalent Equations

$$\frac{1}{2}x + 3 = 13$$
$$\frac{1}{2}x = 10$$
$$x = 20$$

We'll define this in a minute.

- Those three equations are **equivalent equations**.
  - For example, $\frac{1}{2}x + 3 = 3$ is equivalent to $x = 20$.
- Wait… Is "$x = 20$" a solution or an equation?
  - It's both.

Extraneous Solutions

$$\sqrt{5-x} = x-3$$
$$\Rightarrow 5-x = (x-3)^2$$

It would be wrong to write:

$$\sqrt{5-x} = x-3$$
$$\Leftrightarrow 5-x = (x-3)^2$$

- What does it mean for two equations to be equivalent?
- Why have we lost equivalence?
- How do you describe that first step?

Extraneous Solutions

$$\sqrt{5-x} = x-3$$
$$\Rightarrow 5-x = (x-3)^2$$

It would be wrong to write:

$$\sqrt{5-x} = x-3$$
$$\Leftrightarrow 5-x = (x-3)^2$$

Why aren't $5-x=(x-3)^2$ and $\sqrt{5-x}=x-3$ equivalent equations?

- Because they have different solution sets.
- What happened? (We'll answer this shortly.)

Equivalent Equations

- Usiskin (2002, p. 142):"Equivalent equations are those that have exactly the same solutions".
- Sultan and Artzt (2011, p. 273): "When two equations have the same solution set we say they are equivalent."
  - "If the steps used in solving an equation are reversible, then the solution of the final equation and the solution of the original equation are the same. That is, the equations are equivalent. We need not check our answers, though we should."

Usiskin, Z., Peressini, A. L., & Marchisotto, E. (2002). *Mathematics for high school teachers: An advanced perspective*. Prentice Hall.
Sultan, A., & Artzt, A. F. (2011). *The Mathematics that Every Secondary School Math Teacher Needs to Know*. New York, NY: Routledge.

Equivalent Equations

Usiskin (2002, p. 142):"Equivalent equations are those that have exactly the same solutions".

- Is $x + 1 = 50$ equivalent to $\sqrt{x} - 3 = 4$?
- Are $(x - 2)^2 = 0$ and $x + 3 = 5$ equivalent?
- Are $x(x + 1) = x$ and $x + 1 = 1$ equivalent?

Again, it's useful to think of equations as determining sets.

Why did we lose equivalence?

$$\sqrt{5-x} = x-3$$
$$\Rightarrow 5-x = (x-3)^2$$
$$\Leftrightarrow 5-x = x^2-6x+9$$
$$\Leftrightarrow x^2-5x+4=0$$
$$\Leftrightarrow (x-4)(x-1)=0$$
$$\Leftrightarrow x=4; x=1$$

$5-x=(x-3)^2$ and $\sqrt{5-x}=x-3$ are not equivalent - they have different solution sets.

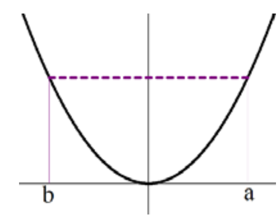
Why did we lose equivalence?

- In the first step, we square both sides.
- $(a^2=b^2) \nRightarrow (a=b)$. For example, $(-3)^2 = 3^2$, but $-3 \neq 3$.
  - It's, of course, true that $(a=b) \Rightarrow (a^2=b^2)$
- $(a^2=b^2) \nRightarrow (a=b)$. In other words, the function $f(x)=x^2$ is not one-to-one.
  - Remember, a function f is one-to-one if and only if $f(a)=f(b) \Rightarrow a=b$. Each output is associated with exactly one input.
  - In fact, since $a=b \Rightarrow f(a)=f(b)$ (functions are well-defined), we can say $f(a)=f(b) \Leftrightarrow a=b$. (More about this in a bit.)
  - **So the first step can be described as applying the function f to both sides.**
- How can you explain this to a high school student?

A student version

$$\sqrt{5-x} = x - 3$$
$$\Rightarrow 5 - x = (x-3)^2$$

- How can you explain this to a high school student?
  - Can you "give examples showing ***how*** extraneous solutions may arise"?
- Would it be unreasonable to walk them through the logic behind the steps?
  - We start with a statement about the equality of expressions/numbers. But we solve a statement about the equality of the squares of those numbers. **(Why?)** BUT just because the squares of two numbers are equal, it doesn't mean those numbers are equal. For example, $3^2 = (-3)^2$ but $-3 \neq 3$.
  - We did something that was not reversible.
- How often should you explain this to a high school student?



We lost equivalence. How can we be sure there aren't other solutions we haven't found?

$$\sqrt{5-x} = x-3$$
$$\Rightarrow 5-x = (x-3)^2$$
$$\Leftrightarrow 5-x = x^2-6x+9$$
$$\Leftrightarrow x^2-5x+4=0$$
$$\Leftrightarrow (x-4)(x-1)=0$$
$$\Leftrightarrow x=4; x=1$$

- We've lost equivalence. We cannot read the implications from bottom to top. In other words, our solution method does not tell us that, if $x = 4$ or $x = 1$, then the original equation is true.
- We've lost equivalence. **Can we even be certain that the solutions to $\sqrt{5-x} = x-3$ are a subset of $\{1, 4\}$?** Or is there maybe some other solution we missed?

Yes!

- Remember, we start with the assumption that the original equation has a solution.
- If $\sqrt{5-x} = x-3$ has a solution, then that implies $5-x = (x-3)^2$, which equivalent to saying $x \in \{1,4\}$. i.e., $\sqrt{5-x} = x-3 \Rightarrow x \in \{1,4\}$.
- So the solution set of $\sqrt{5-x} = x-3$ is some subset of $\{1,4\}$.

Illustrative Mathematics Project Task

Which of the following equations have the same solution? Give reasons for your answer that do not depend on solving the equations.

1. $x + 3 = 5x - 4$
2. $x - 3 = 5x + 4$
3. $2x + 8 = 5x - 3$
4. $10x + 6 = 2x - 8$
5. $10x - 8 = 2x + 6$
6. $0.3 + \frac{x}{10} = \frac{1}{2}x - 0.4$

"The purpose of this task is to provide an opportunity for students to reason about equivalence of equations. The instruction to give reasons that do not depend on solving the equation is intended to focus attention on the transformation of equations as a deductive step."

https://www.illustrativemathematics.org/illustrations/613

Illustrative Mathematics Project Task

In the equations (a)–(d), the solution x to the equation depends on the constant a. Assuming a is positive, what is the effect of increasing a on the solution? Does it increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

a) $x - a = 0$
b) $ax = 1$
c) $ax = a$
d) $\frac{x}{a} = 1$

https://www.illustrativemathematics.org/illustrations/614

"The purpose of this task is to continue a crucial strand of algebraic reasoning begun at the middle school level (e.g, 6.EE.5). By asking students to reason about solutions without explicitly solving them, we get at the heart of understanding what an equation is and what it means for a number to be a solution to an equation. The equations are intentionally very simple; the point of the task is not to test technique in solving equations, but to encourage students to reason about them. "

Solving Equations as Applying Functions

$$\sqrt{5-x} = x - 3$$
$$\Rightarrow 5 - x = (x-3)^2$$

- When you squared both sides of the equation, you can think about that as plugging both sides into the function that squares its input, $f(w) = w^2$.
  - I'm using a w, instead of x, to avoid confusion with the x in $\sqrt{5-x} = x - 3$.
  - It's important to note that f is not one-to-one.

Solving Equations as Applying Functions

- Even with a simple equation…
  - Step 1: Plug both sides into $f(w) = w - 3$.
  - Step 2: Plug both sides into $g(w) = 2w$.
- Are f and g one-to-one?
  - Yes! So $\frac{1}{2}x + 3 = 13 \Leftrightarrow \frac{1}{2}x = 10 \Leftrightarrow x = 20$. We don't need to check our answers. (Though, it's still a good idea.)
  - Multiply x by $\frac{1}{2}$, then add 3. To undo/reverse: subtract 3, then divide by $\frac{1}{2}$.

$$\frac{1}{2}x + 3 = 13$$
$$\frac{1}{2}x = 10$$
$$x = 20$$

Solving Equations as Applying Functions

- So, if you think about solving equations as applying functions you need to worry about whether
  1. the function is one-to-one
  2. the function you are attempting to “undo” is one-to-one.
     - In other words, can you really “undo” it? Does function have an inverse?
     - e.g., applying the square root function attempting to undo squaring.
  3. the function has domain issues. (e.g., you can’t take the square root or the logarithm of a negative number)
     - We won’t talk about this much, but it’s important to keep in mind.

Undoing

- We can think about the first step as applying the function $f(w) = \sqrt{w}$ to both sides.
- What can you tell me about f?
  - It's one-to-one.
  - It's domain is all real numbers greater than or equal to zero.
  - The square root does not undo squaring!
- Is $(x-3)^2 \geq 0$?
  - Yes. So it's fine to take the square root of $(x-3)^2$. There's no domain issue here.
- Is $\sqrt{(x-3)^2} \geq 0$?
  - Yes. The range of f is non-negative reals.
- Is $x - 3 \geq 0$?
  - We have no idea at this point.
  - If we pursue this solution strategy (taking the square root of both sides), we must (*and will*) consider two cases:
    1. $x - 3 \geq 0$
    2. $x - 3 < 0$

$$(x-3)^2 = 16$$
$$\sqrt{(x-3)^2} = \sqrt{16}$$
$$\ldots$$
$$x = -1; x = 7$$

First, some thoughts on $\sqrt{x^2}$

- The square root function takes any non-negative number as its input. The corresponding output is the **positive** square root.
  - A common misconception: $\sqrt{16} = \pm 4$.
- Does the square root "undo" squaring?
  - No. e.g., $(-4)^2 = 16$, taking the square root of 16 does not undo this.
- Something interesting: Why is this statement true? $\sqrt{x^2} = |x|$
- A clever solution could begin:

$$(x-3)^2 = 16$$
$$\sqrt{(x-3)^2} = \sqrt{16}$$
$$|\mathrm{x}-3| = 4$$
$$\ldots$$

$$\sqrt{x^2} = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Some thoughts on $\sqrt{x^2}$ (cont'd)

A student may write:

$$(x-3)^2 = 16$$
$$\sqrt{(x-3)^2} = \sqrt{16}$$
$$\mathrm{x} - 3 = \pm 4$$

- Where does that "$\pm$" come from?
  - Think about the question we're answering: What are all the values of x which make $(x-3)^2 = 16$ true?
- Would it make more sense to write $\pm(x-3) = 4$?
  - Consider $\sqrt{x^2}$.
    - Case 1: $x < 0$. Then $\sqrt{x^2} = -x$. e.g., $\sqrt{(-3)^2} = -(-3)$
    - Case 2: $x \geq 0$. Then $\sqrt{x^2} = x$.
    - Those two cases can be represented by $\pm x$. By this reasoning, we would write $\pm\,(x-3) = 4$ above.
  - $\sqrt{16} = 4$ is a true statement, so it seems strange to replace $\sqrt{16}$ with ± 4.
    - But we are asking, "What are all the values of x which make this equation true?"
    - Since we know that there are exactly two numbers which you can square to get 16, we could ask, "What are all the values of x which make $x-3$ equal to 4 or -4?"
    - By this reasoning, we could write $x - 3 = \pm 4$.
  - What makes more sense? It depends on which reasoning you employ. But it helps to **be explicit about the reasoning. Don't let students think that $\sqrt{16} = \pm 4$.**

Back to our example… a solution

$$(x-3)^2 = 16$$
$$\sqrt{(x-3)^2} = \sqrt{16}$$
$$case\ 1.\ x-3 \geq 0$$
$$x-3 = 4$$
$$x = 7$$
$$case\ 2.\ x-3 < 0$$
$$If\ x-3 < 0, then\ \sqrt{(x-3)^2} = -(x-3)$$
$$-(x-3) = 4$$
$$x = -1$$

Illustrative Mathematics Project Task

1. Solve the following two equations by isolating the radical on one side and squaring both sides:

a. $\sqrt{2x+1} - 5 = -2$

b. $\sqrt{2x+1} + 5 = 2$

Be sure to check your solutions.

You don't need to solve these… but get a sense for the purpose.

2. If we raise both sides of an equation a power, we sometimes obtain an equation which has more solutions than the original one. (Sometimes the extra solutions are called extraneous solutions.) Which of the following equations result in extraneous solutions when you raise both sides to the indicated power ? Explain.

a. $\sqrt{x} = 5$, square both sides

b. $\sqrt{x} = -5$, square both sides

c. $\sqrt[3]{x} = 5$, cube both sides

d. $\sqrt[3]{x} = -5$, cube both sides

$f(w) = w^2$ is not one-to-one
$g(w) = w^3$ is one to one

3. Create a square root equation similar to the one in part (a) that has an extraneous solution. Show the algebraic steps you would follow to look for a solution, and indicate where the extraneous solution arises.

https://www.illustrativemathematics.org/illustrations/391

We hope that students **look for and make use of structure** (SMP 7) and can immediately see that $\sqrt{2x+1} = -3$ has no solutions and $\sqrt{x} = -5$ has no solutions.

Solving Equations as Applying Functions

- Solving equations as applying functions… do you want to go there with students?
  - That's your decision.
  - **It seems excessive for simple examples.** Focusing on reversibility may be sufficient.
  - It does help us explain why we may lose equivalence when, for example, squaring both sides. But, there may be more student-friendly ways.
  - Writing things like "use cross products" may turn opportunities for reasoning into opaque procedures.

3 Extraneous Solutions

Solve $\frac{x-9}{x^2-9} = \frac{-3}{x-3}$. Check your answer.

$(x-9)(x-3) = -3(x^2-9)$ *Use cross products.*

I'm not a fan of this wording. "Cross product" means something else.

$$\frac{1}{2}x + 3 = 13$$
$$\frac{1}{2}x = 10$$
$$x = 20$$

Back to Our Motivating Example

3 Extraneous Solutions

Solve $\frac{x-9}{x^2-9} = \frac{-3}{x-3}$. Check your answer.

$(x-9)(x-3) = -3(x^2-9)$ *Use cross products.*

- When you "use cross products" you are actually multiplying both sides by $(x^2-9)(x-3)$. Is that allowed?
  - Yes! Under the assumption that $(x^2-9)(x-3) \neq 0$. (i.e., $x \neq \pm 3$).
- I have created three cases, though two cases are pretty boring:
  - (1) $x \neq \pm 3$, (2) $x = 3$, (3) $x = -3$
- Multiplying both sides by that expression, you get.

$$\frac{x-9}{x^2-9}(x^2-9)(x-3) = \frac{-3}{x-3}(x^2-9)(x-3)$$

- **I can only "cancel" those factors BECAUSE I assumed $x \neq \pm 3$.**
- What does "cancel" mean here?
  - You are rewriting $\frac{x^2-9}{x^2-9}$ as 1 and $\frac{x-3}{x-3}$ as 1. This are only true if $x \neq \pm 3$.
  - It's very important for students to understand that. Otherwise, they may be in my calculus course making mistakes like this: $\frac{x^2+1}{x^2+3} = \frac{1}{3}$

Extraneous Solutions

$$\frac{x-9}{x^2-9} = \frac{-3}{x-3}$$

- “When you multiply each side of an equation by the LCD [Least Common Denominator], you may get an extraneous solution.”
- What if you multiplied both sides by $x - 100$? Would you introduce an extraneous solution?
  - I guess, if you ignore the fact (as the textbook does) that you can only multiply both sides by $x - 100$ and maintain equivalence if $x - 100 \neq 0$.
- What the textbook was trying to describe really had nothing to do with the LCD.
  - Though, if you multiply both sides by the LCD (and ignore the assumption that it’s nonzero), there is a chance that you might not get an extraneous solution.

Summing up (so far)

- You can multiply or divide both sides of an equation by any expression and maintain equivalence… under the assumption that the expression is non-zero.
  - You can think about this as applying a function to both sides of an equation.
  - $f(w) = aw$ and $g(w) = w/a$ are one-to-one for all $a \neq 0$.
- If you are applying a function to both sides of an equation, then worry about one-to-oneness and domain.
- Thinking about cases often helps.

Students are also instructed to check solutions for logarithmic equations

$$2\ln(x+1) = \ln(1-2x)$$
$$\ln(x+1)^2 = \ln(1-2x)$$
$$e^{\ln(x+1)^2} = e^{\ln(1-2x)}$$
$$(x+1)^2 = 1-2x$$
$$x^2+2x+1 = 1-2x$$
$$x^2+4x = 0$$
$$x(x+4) = 0$$

$$x = 0 \text{ or } x = -4$$

- -4 is not a solution. What happened?

$$\log A^r = r\log A\,, A \geq 0$$
$$x+1 > 0$$

What do you think of this solution?

- A student solves the equation $x(x-5) = x(x-5)(x-1)$ as shown below.

$$x(x-5) = x(x-5)(x-1)$$

$$\frac{x(x-5)}{x-5} = \frac{x(x-5)(x-1)}{x-5}$$

$$x = x(x-1)$$

$$x = x^2 - x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0, x = 2$$

... and when you divide by an expression you must assume that it's not zero.

A student solves the equation $x(x-5) = x(x-5)(x-1)$ as shown below. What went wrong?

$$x(x-5) = x(x-5)(x-1)$$

$$\frac{x(x-5)}{x-5} = \frac{x(x-5)(x-1)}{x-5}$$

$$x = x(x-1)$$

$$x = x^2 - x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0, x = 2$$

Multiplying or dividing both sides of an equation by any nonzero number will yield an equivalent equation.

- Remember the question we are answering: What are all the values of x which make this true?
- You can only divide both sides by $(x-5)$ if you assume $x - 5 \neq 0$. **You must examine two cases: (1) $x \neq 5$ (2) $x = 5$**
- The reason why you can rewrite $\frac{x-5}{x-5}$ as 1 is because you are working under the assumption that $x \neq 5$. In other words, $\frac{x-5}{x-5} = 1$ only for case (1) (only if $x \neq 5$).

Illustrative Mathematics Project Task

Megan is working solving the equation

$$\frac{2}{x^2 - 1} - \frac{1}{x - 1} = \frac{1}{x + 1}$$

She says:

> "If I clear the denominators I find that the only solution is $x = 1$ but when I substitute in $x = 1$ the equation does not make any sense."

1. Is Megan's work correct?
2. Why does Megan's method produce an x value that does not solve the equation?

https://www.illustrativemathematics.org/illustrations/1927

More Equivalent Equations Tasks

From Sultan & Artzt pg. 268-272

1. Jason solves the equations $x^2 = 3x$ by dividing both sides by x to get $x = 3$. He has lost the solution $x = 0$. What did he do wrong?
2. Chan has the equation $\sqrt{x} = -5$ and tries to solve it by squaring both sides. He gets $x = 25$. Yet, when he checks the solution, he realizes it doesn't work since the square root of 25 is positive.
3. Indira solves the equation $(x + 4)(x - 3) = 8$ by setting $x + 4 = 8$ and $x - 3 = 1$ thereby getting the solution $x = 4$ from both equations. She checks her answer by substituting $x = 4$ into the original equation and finds that it works. She concludes that this quadratic equation has only one solution,$x = 4$. But, if we check $x = -5$, it also works. What went wrong?
4. Debbie solves the equation $\frac{x-1}{x+1} = \frac{x-1}{x+3}$ by dividing both sides by $x - 1$ and gets the equation $\frac{1}{x-1} = \frac{1}{x+3}$ from which she concludes by cross multiplying that $x + 1 = x + 3$. She then subtracts x from both sides of this last equations and gets the contradiction that $1 = 3$. She says, "this is impossible," and from this she concludes there is no solution to the original equation. Yet, the original equation has the solution $x = 1$. Where did it go?
5. Are there any values of x for which $\frac{x}{x+1} + \frac{1}{x+1} \neq 1$? If so, what are they?
6. Juan solves the equation $x^2 = 9$ by taking logarithm of both sides. He gets $\log x^2 = \log 9$ by taking the logarithm of both sides. He gets $\log x^2 = \log 3^2$. He remembers that, with logarithms, you can pull the exponent out, so he gets $2 \log x = 2 \log 3$. He divides by 2 to get $\log x = \log 3$, and then concludes that $x = 3$. Yet he missed the solution $x = -3$. Where did it go?

Sultan, A., & Artzt, A. F. (2011). *The Mathematics that Every Secondary School Math Teacher Needs to Know*. New York, NY: Routledge.

Indira

- Indira solves the equation $(x+4)(x-3)=8$ by setting $x+4=8$ and $x-3=1$ thereby getting the solution $x=4$ from both equations. She checks her answer by substituting $x=4$ into the original equation and finds that it works. She conclude that this quadratic equation has only one solution,$x=4$. But, if we check $x=-5$, it also works. What went wrong?
  - I have seen Calc III students make this same mistake. What don't they understand?
  - I've also seen them do this: $(x+4)(x-3)=8 \Rightarrow x+4=8, x-3=8$
  - Semi-related: What does it mean if a student solves $(x+4)(x-3)=0$, writes the solution "$x=3,-4$", and then, upon being prompted to check his solution, writes "$(-4+4)(3-3)=0$"?
  - It takes very little time to emphasize the zero product property.

Solving Equations – A few notes about showing work

- Students often write $\left(\frac{1}{2}x = 6\right) \times 2$ to show that they are multiplying both sides of the equation by 2. What is wrong with that?
- Students often write equal signs between the steps when they are solving an equation. What is wrong with that?

The logic of solving systems of equations

Systems of Equations

- ***Conceptual Category: Algebra, Domain: Reasoning with Equations and Inequalities***
- **CLUSTER: SOLVE SYSTEMS OF EQUATIONS**
  - A-REI.5. Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

But first, try this…

Solve this by setting up a system of equations and then replacing one equation by the sum of that equation and a multiple of the other. Discuss why/how that produces a system with the same solutions.

- A farmer has pigs and chickens. She counts 169 heads and 398 feet. How many pigs and chickens does she have?

Did you notice that this standard did not say "system of two *linear* equations"?

CCSSM A-REI.5

$$\begin{cases} C + P = 169 \\ 2C + 4P = 398 \end{cases}$$

$$\begin{cases} C + P = 169 \\ 2P = 60 \end{cases}$$

- Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
  - This is often called the elimination method. It is one of the 3 elementary row operations when using "Gaussian elimination"
    - Interchange 2 rows
    - Multiply a row by a constant
    - Add a multiple of a row to another row
  - "… produces a system with the same solutions." In other words, we get an equivalent system.
  - Think about a system as a compound sentence: $C + P = 169\ and\ 2C + 4P = 398$.
  - In the work shown above, $2p = 60$ is not equivalent to $2C + 4P = 398$. But the systems are equivalent. $C + P = 169\ and\ 2C + 4P = 398$ is equivalent to $C + P = 169\ and\ 2P$ =60.

Can you translate these into solving systems of linear equations without matrices?

"Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions."

A justification of the elimination method:

- Important things to keep in mind
  - We begin with the assumption that both equations are true
  - the two equations define 2 different relationships between x & y
- "a multiple of the other"… multiply both sides of equation 1 by a non-zero number and get an equivalent equation/relationship. The left hand side and the right hand side represent two ways of writing the same number.
- "replacing one equation by the sum of that equation and a multiple of the other"… add the same number (written two different ways) to both sides of equation 2 and you get an equivalent equation (you are assuming both equations are true).
- "produces a system with the same solutions"… we replaced one equation with an equivalent equation. We get an equivalent system.

Start from the assumption that $C + P = 169$ *and* $2C + 24P = 398$. *Then* $C + P = 169 \Leftrightarrow -2C - 2P = -338$.

$$\begin{cases} C + P = 169 \\ 2C + 4P = 398 \end{cases}$$

$$\begin{cases} C + P = 169 \\ 2C + 4P + (-2C - 2P) = 398 + (-338) \end{cases}$$

$$\begin{cases} C + P = 169 \\ 2P = 60 \end{cases}$$

I added the same thing to each side of the equation. Under our assumption, $-2C - 2P$ is just another way of writing -338.

Does this remind you of A-REI.A.1?

- Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, **starting from the assumption that the original equation has a solution**. Construct a viable argument to justify a solution method.

How to write it.

There's a difference between showing this work….

$$\begin{cases} C + P = 169 \\ 2C + 4P = 398 \end{cases}$$

$$\begin{cases} C + P = 169 \\ 2P = 60 \end{cases}$$

$$\begin{cases} C + P = 169 \\ P = 30 \end{cases}$$

$$\begin{cases} C + P = 169 \\ P = 30 \end{cases}$$

$$\begin{cases} C = 139 \\ P = 30 \end{cases}$$

- In the last step, we also replaced one equation by the sum of that equation and a multiple of the other. Do you see how?
- This is just like Gaussian Elimination. Do you see how?

… and showing this work.

$$C + P = 169$$
$$2C + 4P = 398$$
$$2P = 60$$
$$P = 30$$
$$C + 30 = 169$$
$$C = 139$$

This is the typical way of writing the work.

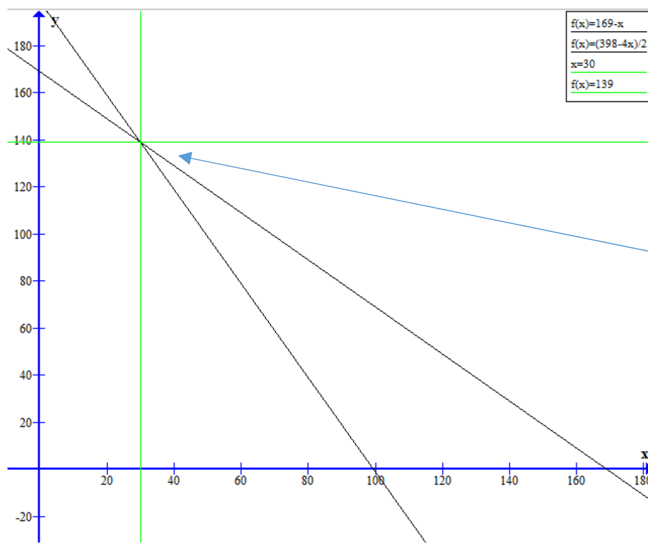
On the left, at each step we are producing an equivalent system. On the right side, we go from having a compound sentence to a bunch of simple sentences. It's harder to follow the logic.

$$\begin{bmatrix} 1 & 1 & 169 \\ 2 & 4 & 398 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 169 \\ 0 & 2 & 60 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 169 \\ 0 & 1 & 30 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 139 \\ 0 & 1 & 398 \end{bmatrix}$$

Solving Graphically.

- One way to solve a system of two equations in two variables is to graph and look for point(s) of intersection.
- The black lines correspond to system 1. The green to system 5. What doesn't change?

1. $\begin{cases} C + P = 169 \\ 2C + 4P = 398 \end{cases}$
2. $\begin{cases} C + P = 169 \\ 2P = 60 \end{cases}$
3. $\begin{cases} C + P = 169 \\ P = 30 \end{cases}$
4. $\begin{cases} C + P = 169 \\ P = 30 \end{cases}$
5. $\begin{cases} C = 139 \\ P = 30 \end{cases}$



The "and" matters!

When does a system have a solution?

- If more equations than unknowns…
  - Generally, no solutions
  - E.g., It's rare for 3 lines to intersect at one point
- If more unknowns than equations…
  - In general, many solutions.
  - E.g., Two planes will usually intersect in a line. (That's an infinite number of solutions.)
- If the same number of equations and unknowns…
  - A lot of machinery in college-level linear algebra deals with this case

SBAC 8

Joe solved this linear system correctly.

$$6x + 3y = 6$$
$$y = -2x + 2$$

These are the last two steps of his work.

$$6x - 6x + 6 = 6$$
$$6 = 6$$

Which statement about this linear system must be true?

A. x must equal 6

B. y must equal 6

C. There are no solutions to this system.

D. There are infinitely many solutions to this system

Think about the compound sentences.

$$\begin{cases} 6x + 3y = 6 \\ y = -2x + 2 \end{cases}$$

$$\begin{cases} 6 = 6 \\ y = -2x + 2 \end{cases}$$

$6 = 6 \; and \; y = -2x + 2$ for any point that satisfies the second equation.

SBAC 8 Extended

- What would the "last step" look like in a system with no solutions. Explain ***why***.

- Is this, in any way, connected?
    - A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
    - We start with the assumption that the system has a solution. We reach a contradiction, so our assumption was wrong.

IMP Grade 8

Consider the equation $5x - 2y = 3$. If possible, find a second linear equation to create a system of equations that has:

- Exactly 1 solution.
- Exactly 2 solutions.
- No solutions.
- Infinitely many solutions.

- Bonus Question: In each case, how many such equations can you find?

https://www.illustrativemathematics.org/

Final Thoughts

- What we discussed today extends to the logic of solving inequalities and systems of inequalities.
- Textbooks haven't done a great job with this.
- It's better to focus on holistic ideas and extensible tools* than on collections of special-purpose tools or rules that apply to one section of a textbook.

*The phrase "extensible tools" borrowed from: Cuoco, A. (2008), Introducing extensible tools in middle-and high-school algebra. *Algebra and Algebraic Thinking: NCTM Yearbook. Reston, VA: NCTM*.

Strongly Disagree	Disagree	Agree	Strongly Agree
0	1	2	3

Send your text message to this Phone Number: 37607

5 digit poll code for this session

44286 *(1 space)* ___ ___ ___ *(no spaces)* *(1 space)* ______________

Speaker was engaging and an effective presenter (0-3)

Speaker was well-prepared and knowledgeable (0-3)

Session matched title and description in program book (0-3)

Other comments, suggestions, or feedback (words)

Thank you!

josh.chesler@csulb.edu

Example: 44286 333 Great session!

Non-Example: XXXXX 3 2 3 Great session!

Non-Example: XXXXX3-2-3Great session!