Discrete Mathematics in High School

Brian Hopkins, Saint Peter’s University, New Jersey

California Mathematics Council - South Annual Conference
Palm Springs, Saturday 7 November 2015
Session overview:

- What is discrete mathematics?
- Standard discrete mathematics topics
- College and high school discrete mathematics course
- Some nonstandard discrete mathematics topics
- Resources
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Interspersed with activities!
**Activity**

**Take 1 or 2** (with 9 coins)
- **Setup:** 9 coins
- **Legal moves:** in each turn, take 1 or 2 coins
- **Winner:** takes the last coin

Any objects will do, or you can draw circles and cross them out. Can the first or second player always win?
Activity

Take 1 or 2 (with 9 coins)

- Setup: 9 coins
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Any objects will do, or you can draw circles and cross them out. Can the first or second player always win?

Variations:

- Different initial number of coins (try 10 next)
- Different numbers of coins can be taken
- Whoever takes last coin loses (misère version)
A version of this was used in the 2002 *Survivor Thailand*

**Take 1, 2, or 3**

- Setup: 21 flags
- Legal moves: in each turn, take 1, 2, or 3 flags
- Winner: takes the last flag

http://www.criticalcommons.org/Members/JJWooten/clips/survivor-21-flags

No one seems to demonstrate any strategy until six flags remain.
What is discrete mathematics?

Negative definition: not continuous.

Often concerned with the integers rather than the real numbers.

Wide array of topics. At the college level, one or both of the following motivations:

- The mathematics needed for computer science
- Proofs course/introduction to advanced mathematics
Discrete Mathematics Topics

- **Set theory & logic**
  - principle of inclusion/exclusion (generalized Venn diagrams)
  - and, or, not; truth tables; circuits
  - for every, there exists, such that

- **Number theory**
  - bases (e.g., binary, hexadecimal)
  - modular systems (clock arithmetic)
  - primes and factorization

Application: RSA cryptography (Rivest, Shamir, Adelman).

The system used on the internet to protect data is secure to the extent that it is computationally hard to factor large numbers. (Look out for quantum computers.)
Methods of proof

- direct
  Every $x$ is $y$. Requires being able to work with an *arbitrary* $x$, not just particular ones. “There is no proof by example.”
- counterexample
  To prove “not every $x$ is $y$,” suffices to find just one $x$ that isn’t $y$.
- multiple cases
- conditional (if then) and contrapositive
  “$p$ implies $q$” is logically equivalent to “not $q$ implies not $p$”
- existence/uniqueness versus constructive
- induction
  Metaphor: Correctly set up a line of dominos then push over the first one.
Suppose a country only has 5 and 7 cent coins. With an unlimited supply of both kinds, what total number of cents can you produce?
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What happens if they start to mint 11 cent coins too?
The largest number that cannot be made from certain values of coins is called the Frobenius number. For values $a$ and $b$, it is $ab - a - b$.

For three or more values, there is no such formula.
Discrete Mathematics Topics

- Recursion and integer sequences
  - Difference tables, polynomials, deceptive patterns
    $(1, 2, 4, 8, 16, 31, 57, \ldots)$, Online Encyclopedia of Integer Sequences, http://oeis.org/A000127.

- Counting
  - Combinations and permutations; does order matter and is repetition allowed; situations beyond those tools.

- Relations and functions
  - Defined on arbitrary sets, not just real numbers.
  - One-to-one, onto, inverse. Relations more general; symmetric, reflexive, transitive, partial orders.

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Discrete Mathematics in High School
Machine learning example

Decision tree for spam detection; searching for which word first leads to more accurate results more quickly?
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Discrete Mathematics in High School
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**weighted average**

$$3.005, 2.987, 3.568$$
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https://oracleofbacon.org

How does the cinema graph change over time?

Find actors farther away (and learn something from your students).

More applications

The 100,000 internet service providers of the web circa 1999.
A discrete mathematics course at the college level would include some of these topics, usually with greater emphasis on proofs or graph theory, going on to algorithms and measure of efficiency for some computer science emphasis.

The typical high school discrete mathematics course?
A discrete mathematics course at the college level would include some of these topics, usually with greater emphasis on proofs or graph theory, going on to algorithms and measure of efficiency for some computer science emphasis.

The typical high school discrete mathematics course? No such thing.

Overlap with some topics in the new AP Computer Science Principles test (fall 2016, first since statistics).
Integer compositions

How many ways can you write $n$ as a sum of positive integers with one of the following restrictions?

A  Only using 1 and 2 as summands.
α  Only using odd numbers as summands.
1  Using every positive integer except 1 as summands.

Pick a restriction and work out the allowed compositions of 1, of 2, up to 7 or 8, looking for patterns.

For this activity, “order matters” so that we will count $1 + 4$ and $4 + 1$ as different compositions.
All number of compositions of $n$ under each destruction is given by the Fibonacci numbers, with different offsets at the beginning.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... 

Each term is the sum of the two previous ones.

How can we prove that all of the compositions of $n$ with only odd part summands are counted by the Fibonacci numbers? Show how to build them from the same compositions of $n − 1$ and $n − 2$. Much more fun to do with Cuisenaire rods than written sums.
Restricted Trains: Just 1s and 2s
Restricted Trains: Just 1s and 2s

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Restricted Trains: Just Odds

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Restricted Trains: Just Odds

Brian Hopkins, bhopkins@saintpeters.edu
Discrete Mathematics in High School
Restricted Trains: No 1s
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Discrete Mathematics in High School
Mathematics of Fair Division

Topic covered in the discrete mathematics course of Michael Stern, Montgomery Township High School, New Jersey.

Historically, applied mathematics has meant applications to physics, economics, computer science, biology. More recently there are a growing number of applications to the social sciences through voting theory, networks, game theory, fair division.

Fair division often is over a continuous domain (land, “cake”) but there are discrete versions:
Suppose Luis & Rita have to split a collection of six plates. They will take turns selecting plates with Luis going first, so that each will end up with three plates.

What makes this interesting is that they may have different preferences.
Let their preferences be these lists, starting with their favorites.

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- What selection procedure do they use to get the best possible collection of three plates?
- How does depend on whether they know each others’ preferences?
Suppose they do not know each other’s preference. Then, at each turn, the player chooses his or her favorite of the available plates.
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Suppose they do not know each other’s preference. Then, at each turn, the player chooses his or her favorite of the available plates.
Now suppose they do know each other’s preferences. How can they use this knowledge?

Rita will end up with the brown plate; might as well take it last and try to do better with her earlier turns.


Repeating this “bottom-up” approach gives the optimal outcome for both players.
Suppose they do know each other’s preference. Work backwards.

Luis

Rita

Luis

Rita
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Suppose they do know each other’s preference. Work backwards.

Luis

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Suppose they do know each other’s preference. Work backwards.
Suppose they do know each other’s preference. Work backwards.
Suppose they do know each other’s preference. Work backwards.
Here, Rita does better with open knowledge, Luis worse.
Here, Rita does better with open knowledge, Luis worse.
(Lower sums are better, Rita $1 + 2 + 6 = 9 > 1 + 2 + 4 = 7$.)
Many related questions, several amenable to elementary exploration.

- In the example, Rita did better in the open knowledge situation. Is that always true? Better for her on average?
- Comparing rank sums implies that the players’ motivation is greed, getting the best possible half based on their preferences. What about other motivations?
Related questions

Many related questions, several amenable to elementary exploration.

- In the example, Rita did better in the open knowledge situation. Is that always true? Better for her on average?
- Comparing rank sums implies that the players’ motivation is greed, getting the best possible half based on their preferences. What about other motivations? Spite, altruism, common good...
- Are there selection orders besides LRLRLR that are fairer? (LRRLLR is better, best is LRLRRL—no nice pattern)
- This connects to a structure called permutations (not the counting term), how does that perspective help?

NCTM’s *Mathematics Teacher* certainly includes discrete mathematics. E.g., article by Anne Quinn this year on using apps for graph theory topics.

Some journals targeted at undergraduate faculty have a lot of overlap with advanced high school topics: *Primus* and the Mathematical Association of America’s *The College Mathematics Journal* (JSTOR).

Navigations series

208 pp., NCTM, 2008, $50.
Standard college textbook


Suggestion: Find an earlier edition; nearly identical, much cheaper.
Resources

Quirky college textbook

Resources

Activities, historical projects, pedagogy articles . . .

Take 1 or 2 adjacent

- Setup: 13 coins in a tight circle (so that adjacent coins touch)
- Legal moves: in each turn, take any one coin or any two touching coins
- Winner: takes the last coin

Evaluate this session: send to 37607, “17404_#_text” where 0 ≤ # ≤ 3 for (prepared, engaging, honest).